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# Parameter estimation for chaotic system with initial random noises by particle swarm optimization ☆

Fei Gao [a,b,*], Ju-Jang Lee [b], Zhuoqiu Li [a], Hengqing Tong [a], Xiaohong Lü [a]

[a] Department of Mathematics, School of Science, Wuhan University of Technology, Luoshi Road 122, Wuhan, Hubei 430070, People's Republic of China
[b] Department of Electrical Engineering and Computer Science, Korean Advanced Institute of Science and Technology, 335 Gwahangno (373-1 Guseong-dong), Yuseong-gu, Daejeon 305-701, Republic of Korea

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ABSTRACT

This paper is concerned with the unknown parameters and time-delays of nonlinear chaotic systems with random initial noises. A scheme based on particle swarm optimization(PSO) is newly introduced to solve the problem via a nonnegative multi-modal nonlinear optimization, which finds a best combination of parameters and time-delays such that an objective function is minimized. The illustrative examples, in chaos systems with time-delays or free, are given to demonstrate the effectiveness of the present method.


## 1. Introduction

Great interests emerged in time-delay chaotic systems [1–9], since the first chaos in time-delay system was discovered by Mackay and Glass [10]. In the these studies, the precise time-delay values of the chaotic system are often assumed to be fully or partially known.

Unfortunately, it is difficult to obtain the exact values of the time delays and the parameters for practical chaotic systems. Actually, parameters identification of chaotic system is the chief task to be resolved and of vital significance.

Many methods have been proposed [11–17] to estimate the unknown parameters of chaos systems without time-delays. Recently, inspired by these methods, Tang put a novel method using Particle swarm intelligence(PSO) [18] to estimate time-delay and parameters for time-delay chaotic systems. However, to the best of authors' knowledge, little research has been done to identify unknown time-delays and parameters of chaotic systems with random initial noises.

PSO is a relatively new computational intelligence tool relation to Evolutionary Computation [19,20], developed by Dr. Eberhart and Dr. Kennedy in 1995 [19]. The dynamics of population in PSO resembles the collective behavior and self-organization of socially intelligent organisms. It has been proved [21] to be a powerful tool for solving optimization problems, especially with non-smooth objective function.

The purpose of this work is to present a simple but effective scheme to identify the unknown parameters and time-delays of nonlinear chaotic systems with random initial noises, such as Lorénz, Lü, Chen systems without time-delay and Logistic chaos system with time-delay. In which, PSO is applied by a proper nonnegative multi-modal numerical optimization problem.

The rest is organized as follows. Section 2 provides brief review for PSO. In Section 3, a proper mathematics model is introduced to transfer the parameter estimation into a multi-modal nonnegative function's optimization. Section 4 includes four

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* Corresponding author. Address: Department of Mathematics, School of Science, Wuhan University of Technology, Luoshi Road 122, Wuhan, Hubei 430070, People's Republic of China. Tel.: +82 42 3508032; fax: +82 42 3505432.
E-mail addresses: hgaofei@gmail.com (F. Gao), jjlee@ee.kaist.ac.kr (J.-J. Lee).
URLs: http://hgaofei.googlepages.com/index.htm (F. Gao), http://iliad.kaist.ac.kr (J.-J. Lee).

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typical simulation examples to illuminate the effectiveness of results obtained. Conclusions are summarized briefly in Section 5.

## 2. The main idea of particle swarm intelligence

PSO belongs to the category of Swarm Intelligence methods closely related to the methods of Evolutionary Computation [13,19–21]. The PSO system is initialized with a population of random solutions and searches for optima by updating generations. The procedure for implementing the global version of PSO is given as following [19,20].

At step $k$, each particle $X_i(k) = (x_{i,1}(k), \dots, x_{i,D}(k))$ $(i = 1, 2, \cdots M)$ keeps track of its coordinates in the feasible space which are associated with the best solution (fitness) it has achieved (self best) so far. (The fitness value is also stored.) This value is called pbest $P_i(k) = (p_{i,1}(k), \dots, p_{i,D}(k))$. When a particle takes all the population as its topological neighbors, the best value is a global best and is called gbest $G(k) = ((g_1(k), \dots, g_D(k))$.

The concept of PSO consists of changing the velocity $V_i(k) = (v_{i,1}(k), \dots, v_{i,D}(k))$ of each particle toward its pbest and gbest locations (PSO without neighborhood model), at each time step. Acceleration is weighted by a random term, with separate random numbers being generated for acceleration toward pbest and gbest locations.

$$\begin{cases} A_{i,d}(k) = r \quad \text{and} \quad (0, c_1) \cdot [p_{i,d}(k) - x_{i,d}(k)] \\ B_{i,d}(k) = r \quad \text{and} \quad (0, c_2) \cdot [g_d(k) - x_{i,d}(k)] \\ v_{i,d}(k+1) = \chi[w \cdot v_{i,d}(k) + A_{i,d}(k) + B_{i,d}(k)] \\ x_{i,d}(k+1) = x_{i,d}(k) + v_{i,d}(k+1) \end{cases} \tag{1}$$

where $w$ is called inertia weight, it often defined as Eq. (2)

$$w := w - k \cdot \frac{w_s - w_e}{T} \tag{2}$$

$w_s, w_e \in [0, 1]$ are the initial and end weight, respectively.

And $c_1, c_2$ are cognition and social acceleration constant respectively, $\chi$ is constriction factor, normally $\chi = 0.9$. $c_1$ determines the effect of the distance between the current position of the particle and its best previous position $P_i$ on its velocity. $c_2$ plays a similar role but it concerns the global best previous position $G(k)$, attained by any particle in the neighborhood. $r$ and $(a, b)$ denotes random in $[a, b]$, in this way, the randomicity is introduced to PSO.

## 3. Mathematics model of the problems

Generally, chaotic systems are described by a set of nonlinear differential equations. In this work, we consider two kinds of chaos systems with initial noises.

### 3.1. Chaos systems without time-delays

Consider the following chaos system

$$\dot{x}(t) = f(x(t), x_0, \Theta) \tag{3}$$

where $x(t) = (x_1(t), \dots, x_n(t))' \in \Re^n$ is the state vector, $x_0 = (x_{10}, \dots, x_{n0})' + rand$ is the initial state vector with random noises, $\Theta = (\theta_1, \dots, \theta_d)'$ is the systematic parameter. The parameter $\Theta$ is to be estimated.

Suppose the structure of system (3) is known, then the estimated system can be written as

$$\dot{\tilde{x}}(t) = f\left(\tilde{x}(t), x_0, \widetilde{\Theta}\right) \tag{4}$$

where $\dot{\tilde{x}}(t) = (\dot{\tilde{x}}_1(t), \dots, \dot{\tilde{x}}_n(t))\prime \in \Re^n$ is the state vector of system (4), $\widetilde{\Theta} = (\tilde{\theta}_1, \dots, \tilde{\theta}_d)'$ is the systematic parameters to be estimated.

Let $L = (x_1, \dots, x_m)$ and $\widetilde{L} = (\tilde{x}_1, \dots, \tilde{x}_m)$, where $m$ denotes the length of sampling time points used for parameter estimation, $\{x_i, \tilde{x}_i (i = 1, \dots, m)\}$ denote the state vector of the original system (8) and the estimated system system (9) at time $i$, respectively.

Then a novel objective function is constructed as:

$$F = \left\| L - \widetilde{L} \right\|^2 \tag{5}$$

where $\widetilde{\Theta}$ is the independent variables of function (5).

We use PSO to minimize the function (5), and Fig. 1 shows the principle of parameter estimation for chaotic systems with random initial noises by the proposed method.

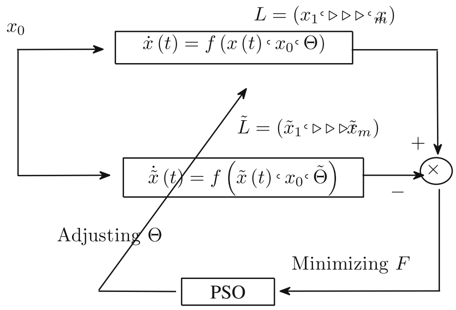



**Fig. 1.** The principle of parameter estimation for chaotic systems with initial noises.

To explain the proposed method, we take the famous Lorénz system [22] for example, the first canonical chaotic attractor found in 1963. Firstly a new system is constructed from Lorénz system:

$$\begin{cases} \dot{x} = \sigma \cdot (y - x); \\ \dot{y} = \gamma \cdot x - x \cdot z - y; \\ \dot{z} = x \cdot y - b \cdot z. \\ L = (x, y, z) \end{cases} \rightarrow \begin{cases} \dot{\tilde{x}} = \sigma \cdot (y - x); \\ \dot{\tilde{y}} = \gamma \cdot x - x \cdot z - y; \\ \dot{\tilde{z}} = x \cdot y - \tilde{b} \cdot z. \\ \tilde{L} = (\tilde{x}, \tilde{y}, \tilde{z}) \end{cases} \tag{6}$$

where $\sigma = 10, \gamma = 28, b$ is unknown. When $b = 8/3$, system (6) is chaotic. And when $b$ is unknown, we have to estimate it.

Secondly, the objective function is chosen as:

$$p^2 = F(\tilde{b}) = \sum_{t=0\cdot h}^{T\cdot h} \left\| \tilde{L} - L \right\|^2 \tag{7}$$

Then the problems of estimation of parameters for chaotic system are transformed into that of nonlinear function optimization. And the smaller the function $f$ is, the better parameter $\tilde{b}$ is.

Thirdly let Lorénz system evolves freely, choose any point as initial point after transience. For a estimated of the $\tilde{b}$, use 4-order Runge–Kutta method to resolve system (6) with $h = 0.01$ to get a discrete time serials of Lorénz system's standard state at $(\tilde{x}(\tilde{b}, t),\ \tilde{y}(\tilde{b}, t),\ \tilde{z}(\tilde{b}, t)), t = 0h, h, \cdots, 300h$.

Let $b_1(i) = 0.1 + 9.9 \cdot i/1000,\ b_2(i) = 0.1 + 9.9 \cdot i/1000, i = 1, \ldots, 1000$, we plot the correspondent objective function (7) for Lorénz system (6) with random initial noises in $[-0.1, 0.1]^3$.

According to Fig. 2, the function (7) for system (6) is multi-modal, whose minimum is near 8/3, but not 8/3. Thus, it is very difficult to get the unknown parameters with the traditional Newton-kind optimization methods. According to Fig. 2, the function (7) is stochastic, even $b_1(i), b_2(i)$ has the same value.

### 3.2. Time-delay chaos systems

Consider the following time-delay chaos system

$$\dot{x}(t) = f(x(t), x(t - \tau), x_0, \Theta) \tag{8}$$

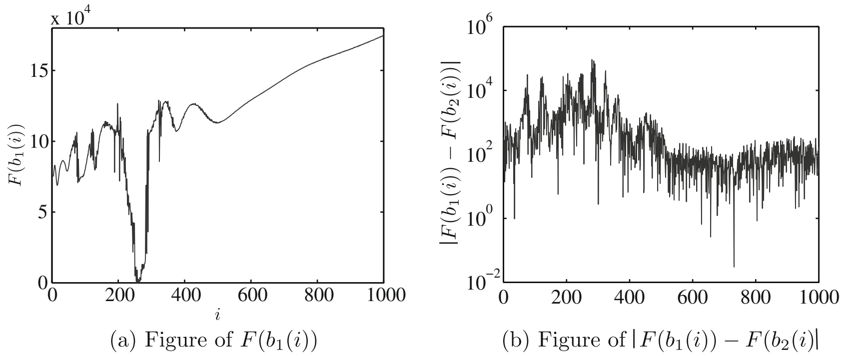



**Fig. 2.** objective function (7) for Lorénz system (6).

where $x(t) = (x_1(t), \ldots, x_n(t))' \in \Re^n$ is the state vector with noises, $x_0 = (x_{10}, \ldots, x_{n0})'$ is the initial state vector, $t > \tau$, $\Theta = (\theta_1, \ldots, \theta_d)'$ is the systematic parameters. And the time-delay $\tau$ is treated as a parameter to be estimated.

Suppose the structure of system (8) is known, then the estimated system can be written as

$$\dot{\tilde{x}}(t) = f\left(\tilde{x}(t), \tilde{x}(t-\tilde{\tau}), x_0, \widetilde{\Theta}\right) \tag{9}$$

where $\dot{\tilde{x}}(t) = (\dot{\tilde{x}}_1(t), \ldots, \dot{\tilde{x}}_n(t))' \in \Re^n$ is the state vector of system (4), $\tilde{\tau}$ and $\widetilde{\Theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_d)'$ is the systematic parameters to be estimated.

DDE23 solver [25] in Matlab7 is chosen to solve system (8) and (9) with an initial state $x_0$ with random initial noises. And 100 states are used. Then $L, \widetilde{L}$ are obtained and the objective function is constructed like Eq. (5).

## 4. Simulations

In this section, firstly the pseudocode of the proposed scheme is given. Secondly, simulations are done to identify the parameters of four benchmark chaos systems with the new method. And all the simulations are done 100 times, respectively.

### 4.1. The novel parameter estimation method

The procedure of the new method is outlined in the following pseudocode:

### 4.2. Simulations to chaos systems with time-delay or free

The first is Lorénz system (6), the second is Chen system [23], which is the dual of the Lorénz system,

$$\begin{cases} x' = \theta_1 \cdot (y - x); \\ y' = (\theta_3 - \theta_1) \cdot x - x \cdot z - \theta_3 \cdot y; \\ z' = x \cdot y - \theta_2 \cdot z. \end{cases} \tag{10}$$

when $\theta_1 = 35, \theta_2 = 3, \theta_3 = 28$.

The third is Lü system [24], which represents the transition between the Lorénz and the Chen attractors,

$$\begin{cases} x' = \theta_1 \cdot (y - x); \\ y' = \theta_3 \cdot y - x \cdot z; \\ z' = x \cdot y - \theta_2 \cdot z. \end{cases} \tag{11}$$

has a chaotic attractor when $\theta_1 = 36, \theta_2 = 3, \theta_3 = 20$.

The fourth system is a time-delay Logistic chaotic system:

$$\dot{x}(t) = -\lambda x(t) + \gamma x(t-\tau)(1 - x(t-\tau)) \tag{12}$$

when $\tau = 0.5,\ \lambda = 26,\ \gamma = 104$, system (12) is chaotic [18] and the chaotic behavior is shown in Fig. 3.

We estimate the $\Theta = (\theta_1, \theta_2, \theta_3)$ of the correspondent systems (10) and (11). And taking $\tau$ in (12) as a unknown parameter, we estimate the $\tau, \lambda, \gamma$ for system (12).

The PSO calculations were done with evolution generation $T = 300$, a population of size $M = 40$. Let $\tilde{b} \in [0.1, 10]$ for Lorénz system. Let $\widetilde{\Theta} \in [30, 40] \times [0, 10] \times [15, 25]$ for Chen system. Let $\widetilde{\Theta} \in [30, 40] \times [0, 10] \times [23, 33]$ for Lü system. 300 states are used here.

For the time-delay logistic chaotic system (12), let $\tilde{\lambda} \in [20, 30], \tilde{\gamma} \in [100, 110], \tilde{\tau} \in [0.05, 1]$. 100 states are used.

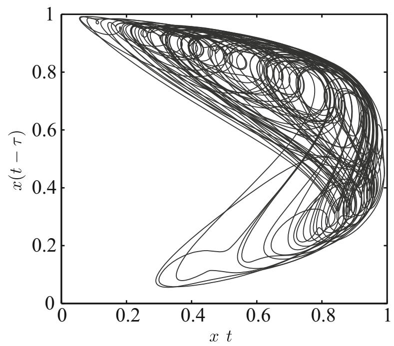
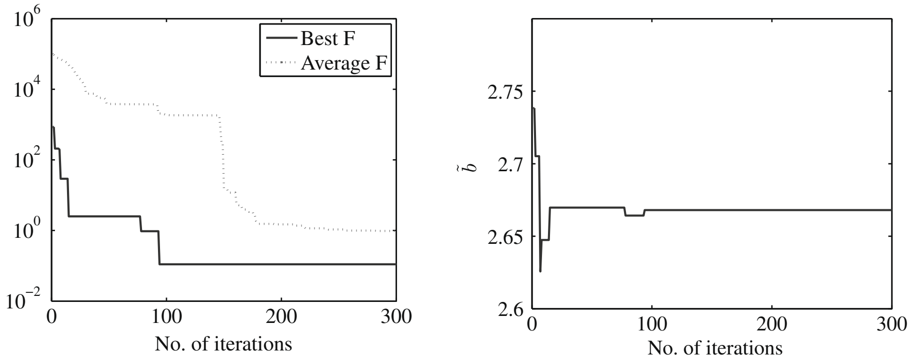
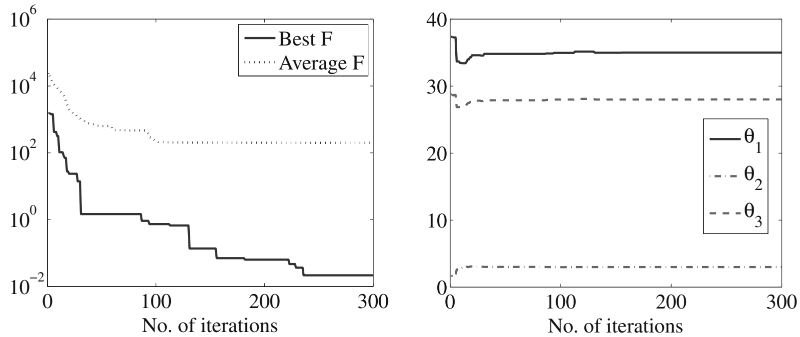



**Fig. 3.** Chaotic behavior of system (12).



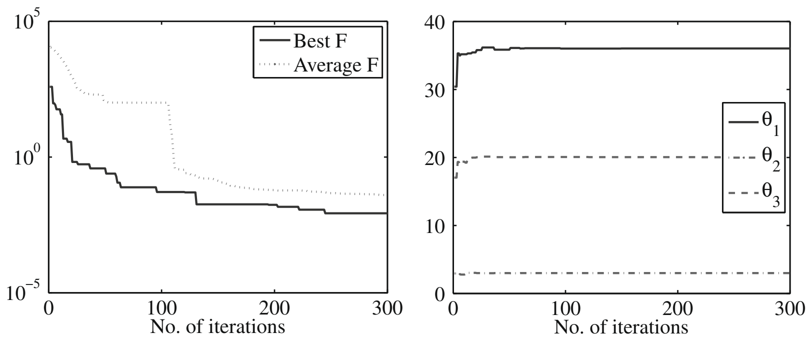

(a) Process of optimizing function (5) (b) Process of optimizing $\tilde{b}$

**Fig. 4.** Simulation for Lorénz system (6).




(a) Process of optimizing function (5) (b) Process of optimizing $\tilde{\Theta}$

**Fig. 5.** Simulation for Chen system (10).




(a) Process of optimizing function (5) (b) Process of optimizing $\tilde{\Theta}$

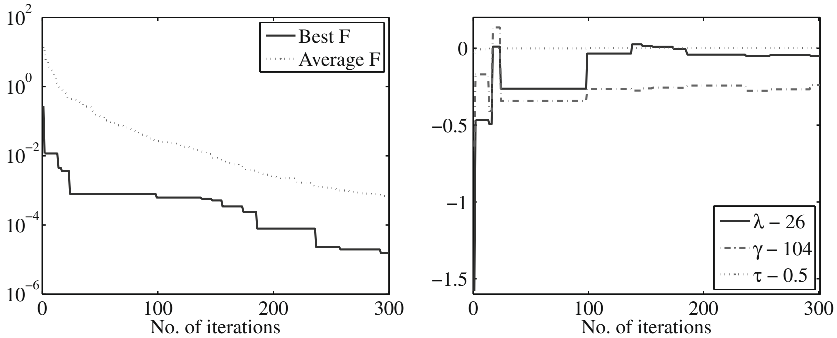
**Fig. 6.** Simulation for Lü system (11).

Figs. 4–7 gives one of the result of the 100 simulations.

According to Figs. 4–7, the PSO converge quickly for chaos system with time-delay or free.

## 5. Conclusions

In this paper, identification of the unknown time-delays and parameters for chaos system is studied based on Algorithm 1 with PSO. Simulation results on chaotic systems free of time delay Lorénz, Chen, Lü systems and Logistic time-delay chaotic system with random initial noises demonstrated the effectiveness and efficiency of PSO.




(a) Process of optimizing function (5) (b) Process of optimizing parameters

**Fig. 7.** Simulation for Logistic system (12).

**Algorithm 1** Parameter estimation for chaos system with PSO

1: **Initialize** Parameters for PSO and chaos system (4);
2: Generate the initial population in the feasible domain of $\widetilde{\Theta}$;
3: **repeat**
4: Optimize the function (5) by PSO
5: **Until** Termination condition is satisfied

The aim of this paper is to design a scheme based on PSO to identify the unknown parameters and time-delays. Though it is not good enough, we hope this method will contribute to the application of chaos control and synchronization. Moreover, it also illustrated the simplicity and easy implementation of the other kind of evolutionary algorithms to replace PSO in the put scheme.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